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TIME ASYMPTOTICS OF STRUCTURED POPULATIONS WITH DIFFUSION AND DYNAMIC BOUNDARY CONDITIONS

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Abstract. This work revisits and extends in various directions a work by J.Z. Farkas and P. Hinow (Math. Biosc and Eng, 8 (2011) 503-513) on structured populations models (with bounded sizes) with diffusion and generalized Wentzell boundary conditions. In particular, we provide first a self-contained L^1 generation theory making explicit the domain of the generator. By using Hopf maximum principle, we show that the semigroup is always irreducible regardless of the reproduction function. By using weak compactness arguments, we show first a stability result of the essential type and then deduce that the semigroup has a spectral gap and consequently the asynchronous exponential growth property. Finally, we show how to extend this theory to models with arbitrary sizes and point out an open problem pertaining to this extension.

1. Introduction.

Structured population models are widely discussed in the literature on population dynamics (see e.g. [17, 19]). A model with size-structure appeared in a work by J.W. Sinko and W. Streifer [30] (see also [36] and the references therein). The introduction of spatial diffusion in population biology goes back to A. Kolmogorov I. Petrovskii and N. Piscunov [16] and J.G. Skellam [31]. We refer to the book by J.D. Murray [23] for a survey of reaction-diffusion equations in biology. Later, R. Waldstätter, K.P. Hadeler and G. Greiner [34] introduced diffusion in structure variable other than space. In [15], K.P. Hadeler introduced diffusion in a size-structured model where the main concern is the understanding of relevant boundary conditions for realistic models. In this context, some special cases of general Robin boundary condition were considered. Other developments for more general boundary conditions are due to J.Z. Farkas and P. Hinow [11], J.Z. Farkas and A. Calsina [6, 7] and A. Bartłomiejczyk and H. Leszczyński [3, 4].

The goal of the present work is to provide a systematic spectral analysis of the diffusive and linear structured population model considered by J.Z. Farkas and P.

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Hinow [11]

$$u_t(s,t) + (\gamma(s)u(s,t))_s = (d(s)u_s(s,t))_s - \mu(s)u(s,t) + \int_0^m \beta(s,y)u(y,t)\mathrm{d}y, \quad (1)$$

with generalized Wentzell-Robin (or dynamic) boundary conditions

$$[(d(s)u_s(s,t))_s]_{s=0} - b_0u_s(0,t) + c_0u(0,t) = 0, \quad (2)$$

$$[(d(s)u_s(s,t))_s]_{s=m} + b_mu_s(m,t) + c_mu(m,t) = 0, \quad (3)$$

and

$$b_0 - \gamma(0) > 0, \quad b_m + \gamma(m) > 0, \quad (4)$$

The different parameters will be defined thereafter. We note that there exists an important literature on second order equations with Wentzell boundary conditions which goes back to W. Feller [13] and A.D. Wentzell [38] (see e.g. A. Favini G.R. Goldstein J.A. Goldstein and S. Romanelli [12] and the references therein). We refer to [4] and to the book by A. Bobrowski [5] Chapter 3 for a biological interpretation of such boundary conditions.

Here $u(s,t)$ denotes the density of individuals of size $s \in [0,m]$ at time $t \geq 0$. The function d stands for the size-specific diffusion coefficient while μ, γ denote respectively the mortality and growth rate of the individuals. Furthermore the non-local integral term in (1) represents the recruitment of individuals into the population. More precisely, $\beta(s,y)$ is the rate at which individuals of size y produce individuals of size s.

The object of this work is to improve and extend [11] *in various directions.*

In [11] the authors write (1)-(2)-(3) in the matrix form

$$\begin{cases} U'(t) &= \mathcal{A}U(t), \\ U(0) &= (u^0, u_0^0, u_m^0) \in \mathcal{X}, \end{cases} \quad (5)$$

where

$$\begin{aligned} \mathcal{A}\begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} &= A\begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} + K\begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} \\ &= \begin{pmatrix} (du')' - (\gamma u)' - \mu u \\ (b_0 - \gamma(0))u'(0) - \rho_0 u_0 \\ -(b_m + \gamma(m))u'(m) - \rho_m u_m \end{pmatrix} + \begin{pmatrix} \int_0^m \beta(.,y)u(y)\mathrm{d}y \\ \int_0^m \beta_0(y)u(y)\mathrm{d}y \\ \int_0^m \beta_m(y)u(y)\mathrm{d}y \end{pmatrix}, \end{aligned}$$

and show their well-posedness in the sense of semigroup theory in the space

$$\mathcal{X} = (L^1(0,m) \times \mathbb{R}^2, \|.\|_{\mathcal{X}})$$

endowed with the norm

$$\|(x, x_0, x_m)\|_{\mathcal{X}} = \|x\|_{L^1(0,m)} + c_1|x_0| + c_2|x_m|$$

where

$$c_1 = \frac{d(0)}{b_0 - \gamma(0)}, \quad c_2 = \frac{d(m)}{b_m + \gamma(m)}.$$

Actually, to deal with well-posedness of the Cauchy problem, the term

$$K\begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} := \begin{pmatrix} \int_0^m \beta(.,y)u(y)\mathrm{d}y \\ \int_0^m \beta_0(y)u(y)\mathrm{d}y \\ \int_0^m \beta_m(y)u(y)\mathrm{d}y \end{pmatrix}$$

can be ignored since it can be treated by elementary (bounded) perturbation arguments. In [11], the authors define first A on *smooth* functions

$$A_s : D(A_s) \to \mathcal{X} \tag{6}$$

where

$$D(A_s) = \{(u, u_0, u_m) \in C^2\,[0,m] \times \mathbb{R}^2 : u(0) = u_0, u(m) = u_m\}$$

and show the *dissipativity* of A_s. Then they refer to C^{α}-theory of elliptic equations ([14] Theorem 6.31) for the proof that the *closure* of A_s denoted by A, is a generator. A priori such an argument gives *no* information on the domain of A apart from the fact that

$$D(A) \supset D(A_s).$$

The authors claim that the generator A is resolvent compact because the embedding of $W^{1,1}[0,m]$ into $L^1(0,m)$ is compact but they do *not* prove that the domain of A is embedded in $W^{1,1}\,[0,m]$. Thus, there is a priori a *gap* in their proof that A is resolvent compact.

Here we define A on an *explicit domain*

$$D(A) = \{(u, u_0, u_m) \in W^{2,1}(0,m) \times \mathbb{R}^2 : u(0) = u_0, u(m) = u_m\}$$

where $W^{2,1}(0,m)$ is the usual Sobolev space of functions in $L^1(0,m)$ having the first two distributional derivatives in $L^1(0,m)$. Indeed, besides dissipativity arguments following [11], we show here directly that the operator is closed, densely defined and satisfies the rank condition. Thus, a *self-contained generation theory with an explicit generator* is given. (In particular, the knowledge of $D(A)$ allows to assert that A is resolvent compact.) This is the *first* contribution of this work.

In [11], the authors show that $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is irreducible under the assumption that β is continuous on $[0,m]^2$ and

$$\beta(.,.) > 0.$$

We show here that this strict positivity assumption is *unnecessary*. Indeed,

$$e^{t\mathcal{A}} \geq e^{tA}$$

and we show that $\left(e^{tA}\right)_{t\geq 0}$ is irreducible by using Hopf's maximum principle. In particular, $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is irreducible even if $\beta = 0$. This is our *second* contribution.

We show the *existence* of an algebraically simple leading real eigenvalue of $\mathcal{A}$. This is our *third* contribution.

We deal also with a much more important issue. Indeed, in [11] the authors "deduce" from the fact that $\mathcal{A}$ is resolvent compact and $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is irreducible that $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ converges (in operator norm) exponentially to the spectral projection P associated to the leading eigenvalue $\widehat{\lambda}$ of $\mathcal{A}$

$$e^{-t\widehat{\lambda}} e^{t\mathcal{A}} \to P \quad (t \to \infty).$$

A priori, such a proof is *not* complete. Indeed, such a conclusion can be reached only if we know that the semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ has a *spectral gap* (i.e. its essential type is *strictly* less than its type) which is *not* at all a consequence of the resolvent compactness of $\mathcal{A}$ and the irreducibility of $e^{t\mathcal{A}}$. In fact, we need to study the spectrum of the semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ *itself*. We can show this property by using tools developped in the context of Transport theory [20, 22]. Indeed, by using weak compactness arguments (we assume that K is weakly compact), we show first that the semigroups $\left(e^{tA}\right)_{t\geq 0}$ and $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ have the same essential type

$$\omega_{ess}(\left(e^{tA}\right)_{t\geq 0}) = \omega_{ess}(\left(e^{t\mathcal{A}}\right)_{t\geq 0});$$

(the weak compactness of K is insured e.g. if there exists $\tilde{\beta} \in L^1(0,m)$ such that

$$\beta(s,y) \leq \tilde{\beta}(s);$$

in particular, it is trivially satisfied if β is continuous on $[0,m]^2$). It follows that the essential type of $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is less than or equal to the spectral bound of A

$$\omega_{ess}(\left(e^{t\mathcal{A}}\right)_{t\geq 0}) \leq s(A) := \sup\left\{\Re(\lambda); \lambda \in \sigma(A)\right\}.$$

Secondly, by exploiting the fact that A is resolvent compact and Marek's results [18], we show that the spectral bound of A is strictly less than that of $\mathcal{A}$

$$s(A) < s(\mathcal{A}) := \sup\left\{\Re(\lambda); \lambda \in \sigma(\mathcal{A})\right\}$$

once

$$K \neq 0$$

i.e. once $\beta(.,.)$ is *not equal to zero* almost everywhere. This implies that $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ exhibits a spectral gap

$$\omega_{ess}(\left(e^{t\mathcal{A}}\right)_{t\geq 0}) < \omega(\left(e^{t\mathcal{A}}\right)_{t\geq 0})$$

where $\omega(\left(e^{t\mathcal{A}}\right)_{t\geq 0})$ is the type of $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ or equivalently the spectral bound of its generator $\mathcal{A}$, i.e.

$$\omega_{ess}(\left(e^{t\mathcal{A}}\right)_{t\geq 0}) < s(\mathcal{A})$$

(the type of a positive semigroup in L^p spaces coincides with the spectral bound of its generator [10] and is an element of the spectrum [20]). The fact that

$$e^{-ts(\mathcal{A})}e^{t\mathcal{A}} \to P \quad (t \to \infty)$$

exponentially is then just a consequence of standard functional analytic results (see e.g. [35] Proposition 2.3). This is our *fourth* (key) contribution.

A *fifth* contribution is the generalization of this theory to the case

$$m = \infty$$

allowing *arbitrary* sizes, i.e. we study also the model

$$u_t(s,t) + (\gamma(s)u(s,t))_s = (d(s)u_s(s,t))_s - \mu(s)u(s,t) + \int_0^\infty \beta(s,y)u(y,t)\mathrm{d}y, \quad (7)$$

$$[(d(s)u_s(s,t))_s]_{s=0} - b_0u_s(0,t) + c_0u(0,t) = 0. \quad (8)$$

To our knowledge, the spectral analysis of this model appears here for the first time. The generation theory in

$$\mathcal{X} = (L^1(0,+\infty) \times \mathbb{R}, \|.\|_{\mathcal{X}})$$

turns out to be much more involved. Indeed, the domain of the generator turns out to be much more tricky since its consists of those $(u,u_0) \in L^1(\mathbb{R}_+) \times \mathbb{R}$ such that

$$u \in W^{2,1}(0,c)\ \forall c > 0, \quad u(0) = u_0$$

$$(du')' - (\gamma u)' \in L^1(\mathbb{R}_+) \ \text{ and } \ \lim_{s\to+\infty} d(s)u'(s) - \gamma(s)u(s) = 0.$$

A priori the domain of the generator is larger than the space

$$\left\{(u,u_0) \in W^{2,1}(\mathbb{R}_+) \times \mathbb{R};\ u(0) = u_0\right\}$$

but we show that this space is a *core* of the domain generator.

As previously, the irreducibility of the semigroup is shown by using Hopf's maximum principle. Similarly, if

$$L^1(\mathbb{R}_+) \ni u \to \int_0^\infty \beta(.,y)u(y)\mathrm{d}y \in L^1(\mathbb{R}_+)$$

is weakly compact, (e.g. if there exists $\tilde{\beta} \in L^1(0,\infty)$ such that

$$\beta(s,y) \leq \tilde{\beta}(s)\ ),$$

then the semigroups $\left(e^{tA}\right)_{t\geq 0}$ and $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ have the same essential type. On the other hand, we *cannot* appeal to Marek's arguments [18] to infer the existence of a spectral gap because $\mathcal{A}$ is *not* a priori resolvent compact. In this case, we show that the spectral gap property

$$\omega_{ess}(\left(e^{t\mathcal{A}}\right)_{t\geq 0}) < s(\mathcal{A})$$

holds if $\beta_0(.) \neq 0$, if there exists a measurable set $I \subset \mathbb{R}_+$ with positive measure such that

$$u \in L^1(\mathbb{R}_+),\ u(y) > 0 \text{ a.e.} \Longrightarrow \int_0^\infty \beta(s,y)u(y)\mathrm{d}y > 0 \text{ a.e } s \in I.$$

and if

$$\lim_{\lambda \to s(A)} r_\sigma(K(\lambda - A)^{-1}) > 1 \tag{9}$$

where r_σ refers to a spectral radius. We do not know whether (9) is always satisfied. In particular, we do not know whether

$$\lim_{\lambda \to s(A)} r_\sigma(K(\lambda - A)^{-1}) = +\infty \tag{10}$$

always holds. Note that if

$$\eta := \lim_{\lambda \to s(A)} r_\sigma(K(\lambda - A)^{-1}) < +\infty$$

then the semigroup generated by

$$A + cK$$

has a spectral gap once

$$c > \eta^{-1}.$$

If β is bounded below by a separable kernel

$$\beta(x,y) \geq \beta_1(x)\beta_2(y) \tag{11}$$

then we show that

$$r_\sigma(K(\lambda - A)^{-1}) \geq \left\| \beta_2 \left((\lambda - A)^{-1} \begin{pmatrix} \beta_1 \\ \beta_1(0) \end{pmatrix} \right)_1 \right\|_{L^1(\mathbb{R}_+)}$$

where $(U)_1$ refers to the first component of $U \in \mathcal{X}$. In particular (9) is satisfied if

$$\lim_{\lambda \to s(A)} \left\| \beta_2 \left((\lambda - A)^{-1} \begin{pmatrix} \beta_1 \\ \beta_1(0) \end{pmatrix} \right)_1 \right\|_{L^1(\mathbb{R}_+)} > 1.$$

Note that (11) holds e.g. if β is continuous at some point $(\overline{x},\overline{y})$ with $\beta(\overline{x},\overline{y}) > 0$. Whether (10) is a general property of such biological models is an open problem.
The authors are indebted to the referees for their constructive remarks and suggestions.

2. **Models with bounded sizes.**

2.1. **Framework and hypotheses.**

In order to analyze the problem described by (1)-(2)-(3), following [11] we rewrite the boundary conditions (2)-(3). We substitute the diffusion term in (2)-(3), by the remainder of (1) evaluated in 0 and m respectively. We thus get the following dynamic equations

$$u_t(0,t) = -u(0,t)\rho_0 + u_s(0,t)(b_0 - \gamma(0)) + \int_0^m \beta_0(y)u(y,t)\mathrm{d}y, \tag{12}$$

$$u_t(m,t) = -u(m,t)\rho_m - u_s(m,t)(b_m + \gamma(m)) + \int_0^m \beta_m(y)u(y,t)\mathrm{d}y, \tag{13}$$

where

$$\rho_0 = \gamma'(0) + \mu(0) + c_0,$$

$$\rho_m = \gamma'(m) + \mu(m) + c_m$$

and

$$\beta_0 = \beta(0,.), \quad \beta_m = \beta(m,.).$$

Following [11], the Banach space

$$\mathcal{X} = (L^1(0,m) \times \mathbb{R}^2, \|.\|_{\mathcal{X}})$$

is endowed with the norm

$$\|(x, x_0, x_m)\|_{\mathcal{X}} = \|x\|_{L^1(0,m)} + c_1|x_0| + c_2|x_m|,$$

where

$$c_1 = \frac{d(0)}{b_0 - \gamma(0)}, \quad c_2 = \frac{d(m)}{b_m + \gamma(m)}.$$

We denote by $\mathcal{X}_+$ the nonnegative cone of $\mathcal{X}$. We introduce some hypotheses on the different parameters:

1. $\gamma, d \in W^{1,\infty}(0,m)$ and $\mu, \beta_0, \beta_m \in L^\infty(0,m)$,
2. the functions μ, γ' and $s \mapsto \beta(s,y)$ are continuous at $s = 0$ and $s = m$ for every $y \in [0,m]$,
3. the operator
$$L^1(0,m) \ni u \to \int_0^m \beta(.,y)u(y)\mathrm{d}y \in L^1(0,m)$$
is weakly compact,
4. $b_0, b_m > 0$, $c_0, c_m \geq 0$, $\beta, \mu \geq 0$ and $d(s) \geq d_0 > 0$ for all $s \in [0,m]$.

Remark 1. According to the general criterion of weak compactness (see e.g. Section 4 in [37]), the third hypothesis amounts to

$$\sup_{y\in[0,m]} \int_0^m \beta(s,y)\mathrm{d}s < \infty \text{ and } \lim_{|E|\to 0} \sup_{y\in[0,m]} \int_E \beta(s,y)\mathrm{d}s = 0$$

and is satisfied as soon as there exists $\tilde{\beta} \in L^1(0,m)$ such that $\beta(s,y) \leq \tilde{\beta}(s)$ a.e. $(s,y) \in [0,m]^2$. This is the case for example if β is continuous on $[0,m]^2$.

Using (1)-(12)-(13), we define the operator $\mathcal{A}$ by:

$$\begin{aligned}
\mathcal{A}\begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} &= A\begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} + K\begin{pmatrix} u \\ u_0 \\ u_m \end{pmatrix} \\
&= \begin{pmatrix} (du')' - (\gamma u)' - \mu u \\ (b_0 - \gamma(0))u'(0) - \rho_0 u_0 \\ -(b_m + \gamma(m))u'(m) - \rho_m u_m \end{pmatrix} + \begin{pmatrix} \int_0^m \beta(.,y)u(y)\mathrm{d}y \\ \int_0^m \beta_0(y)u(y)\mathrm{d}y \\ \int_0^m \beta_m(y)u(y)\mathrm{d}y \end{pmatrix},
\end{aligned}$$

where the domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \{(u, u_0, u_m) \in W^{2,1}(0,m) \times \mathbb{R}^2 : u(0) = u_0, u(m) = u_m\}.$$

We are then concerned with the following Cauchy problem

$$\begin{cases} U'(t) &= \mathcal{A}U(t), \\ U(0) &= (u^0, u_0^0, u_m^0) \in \mathcal{X} \end{cases}$$

where

$$U(t) = (u(t), u_0(t), u_m(t))^T.$$

2.2. **Semigroup generation.**
We show here that $\mathcal{A}$ is the generator of a C_0-semigroup. The dissipativity arguments are essentially those in [11] but we prove directly that $\mathcal{A}$ is closed, densely defined and satisfies the rank condition.

Theorem 2.1. *Let Assumption (4) be satisfied. Then $\mathcal{A}$ is the infinitesimal generator of a quasi-contractive C_0-semigroup $\{U(t)\}_{t\geq 0}$ on $\mathcal{X}$.*

Proof. We may restrict ourselves to the operator A; straightforward (bounded) perturbation arguments will end the proof.

1. Let us show that $\overline{D(A)} = \mathcal{X}$. Let $\left(u, u_0, u_m\right)^T \in \mathcal{X}$. Let $\left(u^j\right)_j$ be C^∞ functions with compact supports such that $u^j \to u$ in $L^1(0,m)$ and

$$\textit{support}\ \left(u^j\right) \subset \left[j^{-1},\ m - j^{-1}\right]$$

We look for a parabola

$$f_0^j(s) = as^2 + bs + c \qquad (s \in \left[0,\ j^{-1}\right])$$

such that

$$f_0^j(0) = u_0, \quad f_0^j(j^{-1}) = 0, \ \frac{d\ f_0^j}{ds}\ (\ j^{-1}) = 0.$$

This amounts to $c = u_0$ and

$$aj^{-2} + b\ j^{-1} + u_0 = 0$$

$$2aj^{-1} + b = 0.$$

We find

$$f_0^j(s) = j^2 u_0 s^2 - 2ju_0 s + u_0 = u_0(js - 1)^2.$$

Similarly, we look for a parabola

$$f_m^j(s) = as^2 + bs + c \qquad (s \in \left[m - j^{-1}, m\right])$$

such that

$$f_m^j(m) = u_m, \quad f_m^j(m - j^{-1}) = 0, \ \frac{d\ f_m^j}{ds}\ (m - j^{-1}) = 0.$$

We find

$$f_m^j(s) = u_m j^2 s^2 - 2u_m j^2 s(m - j^{-1}) + u_m j^2 (m - j^{-1})^2 = u_m j^2 (s - m + j^{-1})^2.$$

Define

$$v^j(s) = \begin{cases} f_0^j(s) \text{ if } s \in \left[0,\ j^{-1}\right] \\ u^j(s) \text{ if } s \in \left[j^{-1},\ m - j^{-1}\right] \\ f_m^j(s) \text{ if } s \in \left[m - j^{-1},\ m\right]. \end{cases}$$

Then $v^j \in W^{2,1}(0,m)$, $v^j(0) = u_0$ and $v^j(m) = u_m$, i.e.

$$\left(v^j, v^j(0), v^j(m)\right)^T \in D(A).$$

Let us show that $v^j \to u$ in $L^1(0,m)$. It suffices to show that

$$\int_0^{j^{-1}} \left|f_0^j(s)\right| ds + \int_{m-j^{-1}}^m \left|f_m^j(s)\right| ds \to 0 \quad (j \to +\infty).$$

1 We have

$$\begin{aligned} \int_0^{j^{-1}} \left|f_0^j(s)\right| ds &= |u_0| \int_0^{j^{-1}} (js - 1)^2 ds \\ &= j^2 |u_0| \int_0^{j^{-1}} (s - j^{-1})^2 ds \\ &= \frac{|u_0|}{3j} \to 0 \quad (j \to +\infty). \end{aligned}$$

Similarly

$$\int_{m-j^{-1}}^m \left|f_m^j(s)\right| ds = \frac{|u_m|}{3j} \to 0 \quad (j \to +\infty).$$

Finally

$$\left(v^j, v^j(0), v^j(m)\right)^T \to \left(u, u_0, u_m\right)^T \text{ in } \mathcal{X}$$

2 and $\overline{D(A)} = \mathcal{X}$.

2. Let us show that for ω large enough $A - \omega$ is a dissipative operator. Let $\lambda > 0$, $U = \left(u, u_0, u_m\right)^T \in D(A)$ and $H = ((\lambda + \omega)I - A)U$.
Let $H = \left(h, h_0, h_m\right)^T$. We have to prove that

$$\|H\|_{\mathcal{X}} \geq \lambda \|U\|_{\mathcal{X}}.$$

By definition of H, we have

$$(\lambda + \widehat{\mu}(s))u(s) + (\gamma u)'(s) - (du')'(s) = h(s), s \in (0,m), \tag{14}$$

$$(\lambda + \ \widehat{\rho}_0)u_0 - (b_0 - \gamma(0))u'(0) = h_0, \tag{15}$$

$$(\lambda + \widehat{\rho}_m)u_m + (b_m + \gamma(m))u'(m) = h_m \tag{16}$$

where

$$\widehat{\mu}(s) := \omega + \mu(s), \quad \widehat{\rho}_0 := \omega + \rho_0, \quad \widehat{\rho}_m := \omega + \rho_m.$$

We multiply (14) by $sign(u(s))$, integrate between 0 and m and then multiply (15) and (16) respectively by $sign(u_0)$ and $sign(u_m)$. We get

$$\lambda\|u\|_{L^1} + \int_0^m \widehat{\mu}|u| - \int_0^m (du')' sign(u) + \int_0^m (\gamma u)' sign(u) = \int_0^m h sign(u),$$

$$(\lambda + \ \widehat{\rho}_0)|u_0| - (b_0 - \gamma(0))u'(0) sign(u(0)) = h_0 sign(u(0)),$$

$$(\lambda + \widehat{\rho}_m)|u_m| + (b_m + \gamma(m))u'(m) sign(u(m)) = h_m sign(u(m))$$

which is equivalent to

$$\lambda\|u\|_{L^1}+\int_0^m \widehat{\mu}|u|-\int_0^m (du')'sign(u)+\int_0^m (\gamma u)'sign(u)=\int_0^m hsign(u), \quad (17)$$

$$u'(0)sign(u(0))=\frac{(\lambda+\ \widehat{\rho}_0)|u_0|}{b_0-\gamma(0)}-\frac{h_0 sign(u(0))}{b_0-\gamma(0)}, \quad (18)$$

$$u'(m)sign(u(m))=-\frac{(\lambda+\widehat{\rho}_m)|u_m|}{b_m+\gamma(m)}+\frac{h_m sign(u(m))}{b_m+\gamma(m)}. \quad (19)$$

(a) Any nonempty open set of the real line is a finite or countable union of *disjoints* open intervals (see [2] Theorem 3.11, p. 51) so

$$\{u>0\}=\{s\in(0,m):u(s)>0\}=\underset{i\in\mathbb{N}}{\cup}(a_{i,1},a_{i,2}),$$
$$\{u<0\}=\{s\in(0,m):u(s)<0\}=\underset{i\in\mathbb{N}}{\cup}(b_{i,1},b_{i,2}).$$

Since $u\in W^{1,1}(0,m)\hookrightarrow C([0,m])$ then $\forall i,j\in\mathbb{N}:u(a_{i,1})=0,\ u(a_{i,2})=0,\ u(b_{j,1})=0$ and $u(b_{j,2})=0$ (except possibly at 0 and m). Thus

$$\begin{aligned}&\int_0^m (\gamma u)'sign(u)=\int_{\{u>0\}}(\gamma u)'-\int_{\{u<0\}}(\gamma u)'\\ =\ &\sum_{i\in\mathbb{N}}[\gamma(a_{i,2})u(a_{i,2})-\gamma(a_{i,1})u(a_{i,1})]-\sum_{j\in\mathbb{N}}[\gamma(b_{j,2})u(b_{j,2})-\gamma(b_{j,1})u(b_{j,1})]\\ =\ &\gamma(m)\left|u(m)\right|-\gamma(0)\left|u(0)\right|. \end{aligned} \quad (20)$$

(b) Consider $\int_0^m (du')'(s)sign(u(s))\mathrm{d}s$. Since $u'\in W^{1,1}(0,m)\hookrightarrow C([0,m])$ we have $\forall i,j\in\mathbb{N}:u'(a_{i,2})\le 0, u'(a_{i,1})\ge 0, u'(b_{j,2})\ge 0$ and $u'(b_{j,1})\le 0$ (except possibly at 0 and m). We have

$$\begin{aligned}&\int_0^m (du')'sign(u)=\int_{\{u>0\}}(du')'-\int_{\{u>0\}}(du')'\\ =\ &\sum_{i\in\mathbb{N}}[d(a_{i,2})u'(a_{i,2})-d(a_{i,1})u'(a_{i,1})]-\sum_{j\in\mathbb{N}}[d(b_{j,2})u'(b_{j,2})-d(b_{j,1})u'(b_{j,1})]\\ \le\ &d(m)u'(m)sign(u(m))-d(0)u'(0)sign(u(0)).\end{aligned}$$

Hence

$$\begin{aligned}&\lambda\|u\|_{L^1}+\int\widehat{\mu}|u|+\gamma(m)|u(m)|-\gamma(0)|u(0)|\\ \le\ &d(m)u'(m)sign(u(m))-d(0)u'(0)sign(u(0))+\int hsign(u).\end{aligned}$$

Since

$$d(0)u'(0)sign(u(0))=\frac{d(0)(\lambda+\ \widehat{\rho}_0)|u_0|}{b_0-\gamma(0)}-\frac{d(0)h_0 sign(u(0))}{b_0-\gamma(0)}$$

and

$$d(m)u'(m)sign(u(m))=-\frac{d(m)(\lambda+\widehat{\rho}_m)|u_m|}{b_m+\gamma(m)}+\frac{d(m)h_m sign(u(m))}{b_m+\gamma(m)}$$

then

$$\lambda\|u\|_{L^1}+\left[\gamma(m)+\frac{d(m)(\lambda+\widehat{\rho}_m)}{b_m+\gamma(m)}\right]|u(m)|+\left[-\gamma(0)+\frac{d(0)(\lambda+\widehat{\rho}_0)}{b_0-\gamma(0)}\right]|u(0)|$$

$$+\int\widehat{\mu}|u|$$

$$\leq \frac{d(m)h_m sign(u(m))}{b_m+\gamma(m)}+\frac{d(0)h_0 sign(u(0))}{b_0-\gamma(0)}+\int h sign(u)$$

$$\leq \frac{d(m)\left|h_m\right|}{b_m+\gamma(m)}+\frac{d(0)\left|h_0\right|}{b_0-\gamma(0)}+\|h\|_{L^1}$$

or

$$\lambda\|u\|_{L^1}+\int\widehat{\mu}|u|+\left[\frac{\gamma(m)}{c_2}+(\lambda+\widehat{\rho}_m)\right]c_2|u(m)|+\left[-\frac{\gamma(0)}{c_1}+(\lambda+\widehat{\rho}_0)\right]c_1|u(0)|$$
$$\leq c_2\left|h_m\right|+c_1\left|h_0\right|+\|h\|_{L^1}.$$

Note that if

$$\frac{\gamma(m)}{c_2}+\widehat{\rho}_m\geq 0 \text{ and } -\frac{\gamma(0)}{c_1}+\widehat{\rho}_0\geq 0$$

then

$$\lambda\|u\|_{L^1}+\int\widehat{\mu}|u|+\lambda c_2|u(m)|+\lambda c_1|u(0)|\leq c_2\left|h_m\right|+c_1\left|h_0\right|+\|h\|_{L^1}.$$

But

$$\frac{\gamma(m)}{c_2}+\widehat{\rho}_m=\frac{\gamma(m)(b_m+\gamma(m))}{d(m)}+\gamma'(m)+\mu(m)+c_m+\omega$$

and

$$-\frac{\gamma(0)}{c_1}+\widehat{\rho}_0=-\frac{\gamma(0)(b_0-\gamma(0))}{d(0)}+\gamma'(0)+\mu(0)+c_0+\omega$$

are *nonnegative* for ω large enough. Hence

$$\lambda\|u\|_{L^1}+\int(\mu+\omega)|u|+\lambda c_2|u(m)|+\lambda c_1|u(0)|\leq c_2\left|h_m\right|+c_1\left|h_0\right|+\|h\|_{L^1}$$

and

$$\lambda\|U\|_{\mathcal{X}}\leq\|H\|_{\mathcal{X}}$$

1 for ω large enough..This ends the proof of the dissipativity of $A-\omega$.

3. Let us prove that $(A, D(A))$ is a *closed* operator.
Let $(U^n)_{n\in\mathbb{N}}:=(u^n,u_0^n,u_m^n)_{n\in\mathbb{N}}\subset D(A)$ and let $U:=(u,u_0,u_m)\in\mathcal{X}$ and $G:=(g,g_0,g_m)\in\mathcal{X}$ such that $\lim_{n\to\infty}\|U^n-U\|_{\mathcal{X}}=0$ and $\lim_{n\to\infty}\|AU^n-G\|_{\mathcal{X}}=0$. Note that

$$u^n(0)=u_0^n\to u_0 \text{ and } u^n(m)=u_m^n\to u_m.$$

Since

$$(b_0-\gamma(0))(u^n)'(0)-\rho_0 u^n(0)\to g_0$$

then

$$(u^n)'(0)\to h_0:=\frac{g_0+\rho_0 u_0}{b_0-\gamma(0)}.$$

Similarly

$$-(b_m+\gamma(m))(u^n)'(m)-\rho_m u^n(m)\to g_m$$

and

$$(u^n)'(m)\to h_m:=-\frac{g_m+\rho_m u_m}{b_m+\gamma(m)}.$$

Let

$$f_n := d(u^n)' - \gamma u^n.$$

Since

$$\big(d(u^n)'\big)' - (\gamma u^n)' - \mu u^n \to g$$

then

$$f_n' \to g + \mu u$$

(L^1 convergence) while

$$f_n(0) = d(0)(u^n)'(0) - \gamma(0)u^n(0) \to d(0)h_0 - \gamma(0)u_0$$

so

$$f_n(x) = f_n(0) + \int_0^x f_n'(s)ds \to z(x) := d(0)h_0 - \gamma(0)u_0 + \int_0^x (g + \mu u)(s)ds$$

(L^1 convergence). It follows that

$$(u^n)' \to \frac{z + \gamma u}{d}$$

(L^1 convergence) so $u \in W^{1,1}(0,m)$ and $u^n \to u$ in $W^{1,1}(0,m)$. In particular

$$u(0) = \lim_{n\to\infty} u^n(0) = \lim_{n\to\infty} u_0^n = u_0$$

and

$$u(m) = \lim_{n\to\infty} u^n(m) = \lim_{n\to\infty} u_m^n = u_m.$$

Knowing that $u^n \to u$ in $W^{1,1}(0,m)$, the fact that

$$\big(d(u^n)'\big)' - (\gamma u^n)' - \mu u^n \to g$$

implies that $(u^n)''$ converges in $L^1(0,m)$ so that $u \in W^{2,1}(0,m)$ and $u^n \to u$ in $W^{2,1}(0,m)$. Finally $U \in D(A)$, $G = AU$. This ends the proof of the closedness of A.

4. Let us prove that $(\lambda I - A) : D(A) \to \mathcal{X}$ is a surjective operator for $\lambda > 0$ large enough.
   We consider first a *particular* case

$$H = \big(h, h_0, h_m\big)^T \in L^2(0,m) \times \mathbb{R}^2.$$

We look for $U := \big(u, u_0, u_m\big)^T \in D(A)$ such that $(\lambda I - A)U = H$, i.e.

$$(\lambda + \mu)u - (du')' + (\gamma u)' = h \text{ in } [0,m], \tag{21}$$

$$(\lambda + \rho_0)u_0 - (b_0 - \gamma(0))u'(0) = h_0, \tag{22}$$

$$(\lambda + \rho_m)u_m + (b_m + \gamma(m))u'(m) = h_m. \tag{23}$$

We multiply (21) by $v \in H^1(0,m)$ and integrate between 0 and m to get

$$\lambda \int_0^m uv + \int_0^m \mu uv - \int_0^m (du')'v + \int_0^m (\gamma u)'v = \int_0^m hv.$$

An integration by parts, with (22)-(23) leads to

$$\lambda \int_0^m uv + \int_0^m \mu uv + \int_0^m du'v' - \int_0^m \gamma uv' + K_0 u(0)v(0) + K_m u(m)v(m)$$
$$= \int_0^m hv + c_1 h_0 v(0) + c_2 h_m v(m), \tag{24}$$

where $K_0 = c_1(\lambda + \rho_0) - \gamma(0)$ and $K_m = c_2(\lambda + \rho_m) + \gamma(m)$.
We define the bilinear form

$$a : H^1(0,m) \times H^1(0,m) \to \mathbb{R}$$

by the left hand side and a linear form $L : H^1(0,m) \to \mathbb{R}$ by the right hand side of (24), to get

$$a(u,v) = L(v).$$

Let us check the conditions of Lax-Milgram Theorem. The continuity of a and L are easily obtained by using the trace theory. The inequality

$$2ab \leq \frac{a^2}{\varepsilon^2} + (\varepsilon b)^2 \quad (\forall \varepsilon > 0)$$

implies

$$\int_0^m \gamma u u' \leq \|\gamma\|_{L^\infty} \|u\|_{L^2} \|u'\|_{L^2} \leq \|\gamma\|_{L^\infty} \left(\frac{\|u\|_{L^2}^2}{2\varepsilon^2} + \frac{\varepsilon^2 \|u'\|_{L^2}^2}{2} \right)$$

and consequently

$$\begin{aligned} |a(u,u)| \quad &\geq \quad \left(\lambda - \frac{\|\gamma\|_{L^\infty}}{2\varepsilon^2}\right) \|u\|_{L^2}^2 + \left(d_0 - \frac{\|\gamma\|_{L^\infty} \epsilon^2}{2}\right) \|u'\|_{L^2}^2 \\ &\quad + K_0 u(0)^2 + K_m u(m)^2. \end{aligned}$$

Taking first $\varepsilon > 0$ small enough and then λ large enough, we finally get a coercivity estimate $|a(u,u)| \geq K\|u\|_{H^1}^2$ where $K > 0$ is a constant. By Lax-Milgram Theorem, for every $H \in L^2(0,m) \times \mathbb{R}^2$, there exists a unique $u \in H^1(0,m)$ such that $a(u,v) = L(v)$ for every $v \in H^1(0,m)$. Now, we need to verify that U belongs to $D(A)$, where U is defined by $U := (u, u(0), u(m)) = (u, u_0, u_m)$. For this, we use (24) with $v \in C_c^\infty([0,m])$. Then

$$\left| \int_0^m d u' v' \right| \leq \left(|\lambda| + \|\mu\|_{L^\infty}\right) \|u\|_{L^2} \|v\|_{L^2} + \|\gamma\|_{L^\infty} \left| \int_0^m u v' \right| + \|h\|_{L^2} \|v\|_{L^2}.$$

Since $u \in H^1(0,m)$ then $\left|\int_0^m u v'\right| \leq C\|v\|_{L^2}$. Consequently

$$\left| \int_0^m d u' v' \right| \leq \left[\left(|\lambda| + \|\mu\|_{L^\infty}\right) \|u\|_{L^2} + C\|\gamma\|_{L^\infty} + \|h\|_{L^2} \right] . \|v\|_{L^2} \leq K \|v\|_{L^2}.$$

Thus $du' \in H^1(0,m)$ and $u \in H^2(0,m) \subset W^{2,1}(0,m)$ so $U \in D(A)$.
Now we prove that $(\lambda I - A)U = H$ i.e. (21)-(22)-(23) are satisfied. An integration by parts of (24) with $v \in C_c^\infty(0,m)$ implies (21). Moreover, an integration by parts of (24) with $v \in C^\infty(0,m)$ and $v(0) = 1$, $v(m) = 0$ (respectively $v(0) = 0, v(m) = 1$) gives us (22) (resp. (23)).

We deal now with the *surjectivity* of $(\lambda I - A)$. Let

$$H = (h, h_0, h_m) \in L^1(0,m) \times \mathbb{R}^2.$$

There exists a sequence $(H_n)_{n\geq 0} = (h^n, h_0, h_m) \in L^2(0,m) \times \mathbb{R}^2$ such that $\lim_{n\to\infty} \|H_n - H\|_{\mathcal{X}} = 0$.
We know that $\forall n \geq 0, \exists!\ U_n \in D(A) : (\lambda I - A)U_n = H_n$. In particular $\forall n, m \geq 0, (\lambda I - A)(U_n - U_m) = H_n - H_m$. Using the dissipativity result shown before, we get

$$\|U_n - U_m\|_{\mathcal{X}} \leq C\|H_n - H_m\|_{\mathcal{X}}.$$

It follows that $(U_n)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{X}$. Let U be its limit. Since $AU_n = -H_n + \lambda U_n$ then AU_n converges to $-H + \lambda U$. The *closedness* of A

implies that $U \in D(A)$ and $(\lambda I - A)U = H$ and this ends the proof of the surjectivity.

Thus A generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ by Lumer-Phillips Theorem (see [26] Theorem 4.3, p. 14). Finally, as a bounded perturbation of A, $\mathcal{A}$ generates also a quasi-contraction C_0-semigroup $\{U(t)\}_{t\geq 0}$. □

2.3. **On irreducibility.**

To understand time asymptotics of $\{U(t)\}_{t\geq 0}$, we need to prove a key result related to positivity. We remind first some definitions and results about positive and irreducible operators. We denote by $\langle .,. \rangle$ the duality pairing between $\mathcal{X}$ and $\mathcal{X}'$.

Definition 2.2.

1. For $x \in \mathcal{X}$, the notation $x > 0$ means $x \in \mathcal{X}_+$ and $x \neq 0$.
2. An operator $O \in L(\mathcal{X})$ is said to be positive if it leaves the positive cone $\mathcal{X}_+$ invariant. We note this by $O \geq 0$.
3. A C_0-semigroup $\{Z(t), t \geq 0\}$ on $\mathcal{X}$ is said to be positive if each operator $Z(t)$ is positive.
4. A positive operator $O \in L(\mathcal{X})$ is said to be positivity improving if for any $x > 0$ and $x' > 0$, we have $\langle Ox, x' \rangle > 0$.
5. A positive operator $O \in L(\mathcal{X})$ is said to be irreducible if for any $x > 0$ and $x' > 0$ there exists an integer n such that $\langle O^n x, x' \rangle > 0$.
6. A C_0-semigroup $\{Z(t), t \geq 0\}$ on $\mathcal{X}$ is said to be irreducible if for any $x > 0$ and $x' > 0$ there exists $t > 0$ such that $\langle Z(t)x, x' \rangle > 0$.

We recall that a C_0-semigroup $\{Z(t), t \geq 0\}$ on $\mathcal{X}$ with generator B is positive if and only if, for λ large enough, the resolvent operator $(\lambda I - B)^{-1}$ is positive. We recall also that a C_0-semigroup $\{Z(t), t \geq 0\}$ on $\mathcal{X}$ with generator B is irreducible if, for λ large enough, the resolvent operator $(\lambda I - B)^{-1}$ is positivity improving, (see e.g. [8] p. 165).

Definition 2.3. For a closed operator $B : D(B) \subset \mathcal{X} \to \mathcal{X}$, we denote by $\sigma(B)$ its spectrum and by $s(B)$ its spectral bound defined by

$$s(B) := \begin{cases} \sup\{\Re(\lambda); \lambda \in \sigma(B)\} & \text{if } \sigma(B) \neq \emptyset, \\ -\infty & \text{if } \sigma(B) = \emptyset. \end{cases}$$

The main result of this subsection is:

Theorem 2.4. *The C_0-semigroup $\{U(t)\}_{t\geq 0}$ is irreducible.*

Proof. We have to show that the resolvent $(\lambda I - \mathcal{A})^{-1}$ is positivity improving for large λ. It is easy to see that for large λ

$$\begin{aligned} (\lambda I - \mathcal{A})^{-1} &= (\lambda I - A - K)^{-1} = (\lambda I - A)^{-1} \sum_{n=0}^{\infty} (K(\lambda I - A)^{-1})^n \\ &= (\lambda I - A)^{-1} + (\lambda I - A)^{-1} \sum_{n=1}^{\infty} (K(\lambda I - A)^{-1})^n. \end{aligned}$$

It follows that *if* $(\lambda I - A)^{-1} \geq 0$ then

$$(\lambda I - \mathcal{A})^{-1} \geq (\lambda I - A)^{-1}$$

because K is a positive operator. Hence it suffices to prove that $(\lambda I - A)^{-1}$ is positivity improving.

Let us show *first* that
$$(\lambda I - A)^{-1} \geq 0.$$
Let $U = (\lambda I - A)^{-1} H$ with $H = (h, h_0, h_m) \in \mathcal{X}_+$. Since $C^+([0, m])$ is dense in $L^1_+(0, m)$, we may assume without loss of generality that
$$h \in C^+([0, m]).$$
Thus
$$(\lambda + \mu(s))u(s) + (\gamma u)'(s) - (du')'(s) = h(s), s \in (0, m),$$
$$(\lambda + \ \rho_0)u_0 - (b_0 - \gamma(0))u'(0) = h_0$$
$$(\lambda + \rho_m)u_m + (b_m + \gamma(m))u'(m) = h_m.$$
The first equation is
$$-u'' + \rho_1 u' + \rho_2 u = \rho_3$$
where $\rho_1 = -(d' - \gamma)/d$,
$$\rho_2(s) = (\lambda + \mu(s) + \gamma'(s))/d(s) > 0 \ \forall s \text{ for } \lambda \text{ large enough}$$
and
$$\rho_3 = h/d \geq 0.$$
The *absolute minimum* of u is achieved at some $\overline{s} \in [0, m]$. Let us show that $u(\overline{s}) \geq 0$. If not, i.e. if $u(\overline{s}) < 0$ then $\overline{s} \notin (0, m)$. Indeed, this would imply that
$$0 \geq -u''(\overline{s}) = -\rho_2(\overline{s})u(\overline{s}) + \rho_3(\overline{s}) \geq -\rho_2(\overline{s})u(\overline{s}) > 0$$
which is contradictory. Hence $\overline{s} = 0$ or $\overline{s} = m$. If $\overline{s} = 0$ since
$$(\lambda + \ \rho_0)u(0) - (b_0 - \gamma(0))u'(0) = h_0$$
then
$$-(b_0 - \gamma(0))u'(0) = -(\lambda + \ \rho_0)u(0) + h_0 \geq -(\lambda + \ \rho_0)u(0) > 0.$$
It follows that $u'(0) < 0$ and then $u'(s) < 0$ in the neighborhood of $s = 0$ which contradicts the fact that the absolute minimum is achieved at 0. We argue similarly if $\overline{s} = m$. Finally, $u \geq 0$.

Let us show now that $(\lambda I - A)^{-1}$ is *positivity improving*.
We note first that for any $\mu > \lambda$, the resolvent identity
$$(\lambda I - A)^{-1} = (\mu I - A)^{-1} + (\mu - \lambda)(\lambda I - A)^{-1}(\mu I - A)^{-1}$$
shows that
$$(\lambda I - A)^{-1} \geq (\mu - \lambda)(\lambda I - A)^{-1}(\mu I - A)^{-1}$$
so
$$(\lambda I - A)^{-1} H \geq (\lambda I - A)^{-1} G$$
where
$$G := (\mu - \lambda)(\mu I - A)^{-1} H \in \mathcal{X}_+$$
has the peculiarity of belonging to
$$D(A) \subset W^{2,1}(0, m) \times \mathbb{R}^2 \subset C([0, m]) \times \mathbb{R}^2.$$
Hence, without loss of generality, we may assume that $H = (h, h_0, h_m) \in \mathcal{X}_+$ is such that $h \in C^+([0, m])$. Let us show that
$$u(s) > 0 \text{ a.e.}, \ u(0) > 0, \ u(m) > 0$$
once
$$H = (h, h_0, h_m) \in \mathcal{X}_+ - \{0\}.$$

Let us show by contradiction that $\min u > 0$.

The *absolute minimum* of u is achieved at some $\overline{s} \in [0, m]$. Suppose $u(\overline{s}) = 0$. Then

$$v := -u$$

satisfies the equation

$$v'' - \rho_1 v' + \widetilde{\rho}_2 v = h/d \geq 0$$

where $\widetilde{\rho}_2 \leq 0$. Note that

$$\max v = -\min u \geq 0.$$

If u reaches its minimum in $(0, m)$ then v reaches its maximum in $(0, m)$. By the maximum principle (see [27] Theorem 3, p. 6), v must be constant and then u is equal to the constant $u(\overline{s}) = 0$. It follows that

$$0 = h_0,\ 0 = h_m,\quad 0 = h$$

which is contradictory. Hence

$$u(s) > 0 \ \ \forall s \in (0, m)$$

and $u(0) = 0$ or $u(m) = 0$. Thus v reaches its maximum (equal to zero) at $\overline{s} = 0$ or $\overline{s} = m$. If $\overline{s} = 0$ then $v'(0) < 0$ by Hopf's maximum principle (see [27] Theorem 4, p. 7); since

$$(b_0 - \gamma(0))v'(0) = h_0 \geq 0$$

we get a contradiction. If $\overline{s} = m$ then $v'(m) > 0$ by Hopf's maximum principle; since

$$-(b_m + \gamma(m))u'(m) = h_m$$

we get also a contradiction. Finally $\min u > 0$. □

2.4. On the spectral bound of the generator.

Let $s(\mathcal{A})$ be the spectral bound of $\mathcal{A}$. We have:

Theorem 2.5. *The spectral bound of $\mathcal{A}$ is finite, i.e. $s(\mathcal{A}) > -\infty$.*

Proof. According to Theorem 2.4, for $\lambda > s(\mathcal{A})$, $(\lambda - \mathcal{A})^{-1}$ is positivity improving and therefore irreducible. Since $(\lambda - \mathcal{A})^{-1}$ is also compact then

$$r_\sigma((\lambda - \mathcal{A})^{-1}) > 0,$$

(see [25] Theorem 3), where

$$r_\sigma(O) = \sup\{|\lambda| : \lambda \in \sigma(O)\}$$

is the *spectral radius* of O a bounded operator. On the other hand

$$r_\sigma((\lambda - \mathcal{A})^{-1}) = \frac{1}{\lambda - s(\mathcal{A})}$$

(see [24] Proposition 2.5, p. 67) whence $s(\mathcal{A}) > -\infty$. □

Remark 2. Theorem 2.5 provides us with the existence of a real leading eigenvalue since $s(\mathcal{A}) \in \sigma(\mathcal{A})$ (see e.g. [20] Theorem 5.2, p. 102).

2.5. **On asynchronous exponential growth.**

Let us remind some definitions and results about *asynchronous exponential growth* (see [10], [24] and [35] for the details).

Definition 2.6. Let $\mathcal{L}(\mathcal{X})$ be the space of bounded linear operators on $\mathcal{X}$ and let $\mathcal{K}(\mathcal{X})$ be the subspace of compact operators on $\mathcal{X}$. The essential norm $\|L\|_{ess}$ of $L \in \mathcal{L}(\mathcal{X})$ is given by

$$\|L\|_{ess} = \inf_{K \in \mathcal{K}(\mathcal{X})} \|L - K\|_X.$$

Let $\{Z(t); t \geq 0\}$ be a C_0-semigroup on $\mathcal{X}$ with generator $B : D(B) \subset \mathcal{X} \to \mathcal{X}$. The growth bound (or type) of $\{Z(t); t \geq 0\}$ is given by

$$\omega_0(B) = \lim_{t \to \infty} \frac{\ln(\|Z(t)\|_X)}{t},$$

and the essential growth bound (or essential type) of $\{Z(t); t \geq 0\}$ is given by

$$\omega_{ess}(B) = \lim_{t \to \infty} \frac{\ln(\|Z(t)\|_{ess})}{t}.$$

Definition 2.7 (Asynchronous Exponential Growth)**.** [35, Definition 2.2] Let $\{Z(t)\}_{t\geq 0}$ be a C_0-semigroup with infinitesimal generator B in the Banach space X. We say that $\{Z(t)\}_{t\geq 0}$ has asynchronous exponential growth with intrinsic growth constant $\lambda_0 \in \mathbb{R}$ if there exists a nonzero finite rank operator P_0 in X such that $\lim_{t\to\infty} e^{-\lambda_0 t} Z(t) = P_0$.

We recall the following standard result (see e.g. [8] Theorem 9.11, p. 224).

Theorem 2.8. *Let X be a Banach lattice and let $\{Z(t)\}_{t\geq 0}$ be a positive C_0-semigroup on X with infinitesimal generator B. If $\{Z(t)\}_{t\geq 0}$ is irreducible and if*

$$\omega_{ess}(B) < \omega_0(B)$$

then $\{Z(t)\}_{t\geq 0}$ has asynchronous exponential growth with intrinsic growth constant $\lambda_0 = \omega_0(B)$ and one-rank spectral projection P_0.

Remark 3. Note that $\mathcal{A}$ has a compact resolvent (and consequently the spectrum of $\mathcal{A}$ is composed (at most) of isolated eigenvalues with finite algebraic multiplicity). This follows from the fact that the canonical injection $i : (D(A), \|.\|_{D(A)}) \to (\mathcal{X}, \|.\|_{\mathcal{X}})$ is compact (by Rellich Kondrachov's Theorem) and $D(A) = D(\mathcal{A})$ since $K \in \mathcal{L}(\mathcal{X})$ (see e.g. [10] Proposition II.4.25, p. 117).

We are ready to give the main result of this subsection.

Theorem 2.9. *If $K \neq 0$ then the semigroup $\{U(t)\}_{t\geq 0}$ generated by $\mathcal{A}$ has asynchronous exponential growth.*

Proof. The semigroups $\{U(t)\}_{t\geq 0}$ and $\{T(t)\}_{t\geq 0}$ are related by the Duhamel equation

$$U(t) = T(t) + \int_0^t T(t-s)KU(s)ds.$$

Since K is a weakly compact operator then so is $T(t-s)KU(s)$ for all $s \geq 0$. It follows that the *strong* integral

$$\int_0^t T(t-s)KU(s)ds$$

is a weakly compact operator (see [21] Theorem 1 or [29] Theorem 2.2). Hence $U(t)-T(t)$ is a weakly compact operator and consequently (see [21] Theorem 2.10) $\{U(t)\}_{t\geq 0}$ and $\{T(t)\}_{t\geq 0}$ have the *same* essential type

$$\omega_{ess}(\mathcal{A})=\omega_{ess}(A),$$

in particular

$$\omega_{ess}(\mathcal{A})\leq\omega_0(A).$$

Let $\lambda>s(\mathcal{A})\geq s(A)$. The positivity improving compact operators $O_1:=(\lambda-A)^{-1}$ and $O_2:=(\lambda-\mathcal{A})^{-1}$ are such that

$$O_2\geq O_1\geq 0 \text{ and } O_2\neq O_1$$

since $K\neq 0$. It follows from ([18] Theorem 4.3) that

$$r_\sigma(O_1)<r_\sigma(O_2).$$

In addition, according to ([24] Proposition 2.5, p. 67),

$$r_\sigma\left[(\lambda-A)^{-1}\right]=\frac{1}{\lambda-s(A)} \text{ and } r_\sigma\left[(\lambda-\mathcal{A})^{-1}\right]=\frac{1}{\lambda-s(\mathcal{A})}$$

so

$$s(A)<s(\mathcal{A}).$$

Note that $s(A)=\omega_0(A)$ and $s(\mathcal{A})=\omega_0(\mathcal{A})$ since $\{U(t)\}_{t\geq 0}$ and $\{T(t)\}_{t\geq 0}$ are positive semigroups on L^1 spaces (see e.g. [10] Theorem VI.1.15, p. 358) so $\omega_0(A)<\omega_0(\mathcal{A})$ and

$$\omega_{ess}(\mathcal{A})<\omega_0(\mathcal{A}).$$

By combining this last result and the irreducibility of $\{U(t)\}_{t\geq 0}$, Theorem 2.8 ends the proof. □

Remark 4. Note that in Theorem 2.9, the requirement $K\neq 0$ amounts to the fact that the function β is not identically zero.

3. **Models with unbounded sizes.**

From now on, we consider the general model, described by (7)-(8).

3.1. **Framework and hypotheses.**

The boundary condition (8) can be rewritten into the following dynamic form

$$u_t(0,t)=-u(0,t)\rho_0+u_s(0,t)(b_0-\gamma(0))+\int_0^\infty\beta_0(y)u(y,t)\mathrm{d}y. \tag{25}$$

Let

$$\mathcal{X}_\infty=(L^1(0,\infty)\times\mathbb{R},\|.\|_{\mathcal{X}_\infty})$$

with norm

$$\|(x,x_0)\|_{\mathcal{X}_\infty}=\|x\|_{L^1(0,\infty)}+c_1|x_0|.$$

We assume that

$$b_0-\gamma(0)>0 \tag{26}$$

and denote by $\mathcal{X}_{\infty,+}$ the nonnegative cone of $\mathcal{X}_\infty$. We now introduce some hypotheses on the different parameters:

1. $\gamma,d\in W^{1,\infty}(0,\infty)$ and $\mu,\beta_0\in L^\infty(0,\infty)$,
2. the functions μ,γ' and $s\mapsto\beta(s,y)$ are continuous at $s=0$, for every $y\geq 0$,

3. the operator
$$L^1(0,\infty) \ni u \to \int_0^\infty \beta(.,y)u(y)\mathrm{d}y \in L^1(0,\infty)$$
is weakly compact,
4. $b_0 > 0$, $c_0 \geq 0$, $\beta, \mu \geq 0$ and $d(s) \geq d_0 > 0$ a.e. $s \geq 0$.

Remark 5. According to the general criterion of weak compactness, the third hypothesis amounts to
$$\sup_{y\in[0,\infty)} \int_0^\infty \beta(s,y)\mathrm{d}s < \infty, \quad \lim_{c\to+\infty} \sup_{y\in[0,\infty)} \int_c^\infty \beta(s,y)\mathrm{d}s = 0,$$
$$\lim_{|E|\to 0} \sup_{y\in[0,\infty)} \int_E \beta(s,y)\mathrm{d}s = 0.$$

Define
$$W^{2,1}_{loc}(\mathbb{R}_+) := \left\{u \in L^1_{loc}(\mathbb{R}_+);\ u \in W^{2,1}(0,c)\ \forall c > 0\right\}.$$
By means of (7)-(25), we define the operator $\mathcal{A}_\infty$ by
$$\begin{aligned}\mathcal{A}_\infty \begin{pmatrix} u \\ u_0 \end{pmatrix} &= A_\infty \begin{pmatrix} u \\ u_0 \end{pmatrix} + K_\infty \begin{pmatrix} u \\ u_0 \end{pmatrix} \\ &= \begin{pmatrix} (du')' - (\gamma u)' - \mu u \\ (b_0 - \gamma(0))u'(0) - \rho_0 u_0 \end{pmatrix} + \begin{pmatrix} \int_0^\infty \beta(.,y)u(y)\mathrm{d}y \\ \int_0^\infty \beta_0(y)u(y)\mathrm{d}y \end{pmatrix}\end{aligned}$$
with domain $D(\mathcal{A}_\infty)$ given by
$$\{(u,u_0) \in \mathcal{X}_\infty;\ u \in W^{2,1}_{loc}(\mathbb{R}_+),\ u(0) = u_0,\ \ (du')' - (\gamma u)' \in L^1(\mathbb{R}_+)$$
$$\text{and } \lim_{s\to+\infty} d(s)u'(s) - \gamma(s)u(s) = 0\}.$$
Note that
$$d(s)u'(s) - \gamma(s)u(s) = d(0)u'(0) - \gamma(0)u(0) + \int_0^s z(\tau)d\tau$$
where
$$z := (du')' - (\gamma u)' \in L^1(\mathbb{R}_+)$$
shows that $\lim_{s\to+\infty} d(s)u'(s) - \gamma(s)u(s)$ exists.
As previously, we are concerned with the Cauchy problem
$$\begin{cases} U'(t) &= \mathcal{A}_\infty U(t), \\ U(0) &= (u^0, u^0_0) \in \mathcal{X}_\infty \end{cases}$$
where
$$U(t) = (u(t), u_0(t))^T.$$

3.2. Semigroup generation.

The main result of this subsection is:

Theorem 3.1. *Let Assumption (26) be satisfied. Then $\mathcal{A}_\infty$ is the infinitesimal generator of a quasi-contractive C_0-semigroup $\{U_\infty(t)\}_{t\geq 0}$ on $\mathcal{X}_\infty$.*

Proof. As previously, we restrict ourselves to A_∞ since K_∞ is bounded.

1. Let us show that $\overline{D(A_\infty)} = \mathcal{X}_\infty$. Let $(u,u_0)^T \in \mathcal{X}_\infty$. Let $(u^j)_j$ be C^∞ functions with compact supports such that $u^j \to u$ in $L^1(0,\infty)$ and
$$\text{support } (u^j) \subset \left[j^{-1}, +\infty\right).$$

As in the finite case, we introduce the functions

$$v^j(s) = \begin{cases} f_0^j(s) \text{ if } s \in \left[0,\ j^{-1}\right] \\ u^j(s) \text{ if } s \geq j^{-1}, \end{cases}$$

where

$$f_0^j(s) = j^2 u_0 s^2 - 2ju_0 s + u_0 = u_0(js-1)^2$$

and we verify that

$$D(A_\infty) \ni (v^j, v^j(0))^T \to (u, u_0)^T \in \mathcal{X}_\infty$$

1 so $\overline{D(A_\infty)} = \mathcal{X}_\infty$.

2. Let us prove that $(A_\infty, D(A_\infty))$ is a closed operator. We argue as previously. Let $(U^n)_{n\in\mathbb{N}} := (u^n, u_0^n)_{n\in\mathbb{N}} \subset D(A_\infty)$ then let $U := (u, u_0) \in \mathcal{X}_\infty$ and $G := (g, g_0) \in \mathcal{X}_\infty$ such that $\lim_{n\to\infty} \|U^n - U\|_{\mathcal{X}_\infty} = 0$ and $\lim_{n\to\infty} \|A_\infty U^n - G\|_{\mathcal{X}_\infty} = 0$. Let

$$f_n := d(u^n)' - \gamma u^n.$$

2 Note that by assumption

$$\lim_{s\to+\infty} f_n(s) = 0. \tag{27}$$

Since

$$(d(u^n)')' - (\gamma u^n)' - \mu u^n \to g$$

($L^1(0,\infty)$ convergence) and

$$(b_0 - \gamma(0))\,(u^n)'\,(0) - \rho_0 u^n(0) \to g_0$$

then

$$f_n' \to g + \mu u$$

($L^1(0,\infty)$ convergence) while

$$f_n(0) = d(0)(u^n)'(0) - \gamma(0)u^n(0) \to d(0)h_0 - \gamma(0)u_0$$

where

$$h_0 := \frac{g_0 + \rho_0 u_0}{b_0 - \gamma(0)}.$$

3 Hence

$$f_n(s) = f_n(0) + \int_0^s f_n'(\tau)d\tau \to z(s) := d(0)h_0 - \gamma(0)u_0 + \int_0^s (g+\mu u)(\tau)d\tau \tag{28}$$

in $L^1(0,c)$ for any finite c. It follows that

$$(u^n)' \to \frac{z + \gamma u}{d}$$

in $L^1(0,c)$ for any finite c so $u' \in L^1(0,c)$ and $u^n \to u$ in $W^{1,1}(0,c)$ for any finite c. In particular

$$u(0) = \lim_{n\to\infty} u^n(0) = \lim_{n\to\infty} u_0^n = u_0.$$

Finally

$$f_n' - \mu u^n = (d(u^n)')' - (\gamma u^n)' - \mu u^n \to g$$

($L^1(0,\infty)$ convergence) implies that $(u^n)''$ converges in $L^1(0,c)$ for any finite c so that $u \in W^{2,1}(0,c)$ for any finite c and

$$(d(u)')' - (\gamma u)' - \mu u = g.$$

Note that (28) shows that

$$\begin{aligned} & |f_n(s) - z(s)| \\ \leq \quad & |f_n(0) - (d(0)h_0 - \gamma(0)u_0)| + \int_0^{+\infty} |f_n'(\tau) - (g(\tau) + \mu(\tau)u(\tau))|\, d\tau \to 0 \end{aligned}$$

so

$$f_n(s) \to z(s) = d(s)u'(s) - \gamma(s)u(s) \ \textit{uniformly} \text{ on } \mathbb{R}_+$$

and (27) implies

$$\lim_{s\to+\infty} d(s)u'(s) - \gamma(s)u(s) = 0.$$

1 Thus $U \in D(A_\infty)$ and $G = A_\infty U$.

3. We consider now the dissipativity of $(A_\infty - \omega I)$ for ω large enough. Let $\lambda > 0, U = \left(u, u_0\right)^T \in D(A_\infty)$ and $H = ((\lambda + \omega)I - A_\infty)U$.
Let $H = \left(h, h_0\right)^T$. We have to prove that

$$\|H\|_{\mathcal{X}_\infty} \geq \lambda \|U\|_{\mathcal{X}_\infty}.$$

By definition of H, we have

$$\begin{aligned} (\lambda + \widehat{\mu}(s))u(s) + (\gamma u)'(s) - (du')'(s) = h(s), s \in (0, \infty), \\ (\lambda + \widehat{\rho}_0)u_0 - (b_0 - \gamma(0))u'(0) = h_0, \end{aligned}$$

where

$$\widehat{\mu}(s) := \omega + \mu(s), \quad \widehat{\rho}_0 := \omega + \rho_0.$$

By integration

$$\begin{aligned} &\lambda \|u\|_{L^1} + \int_0^\infty \widehat{\mu}|u| - \int_0^\infty (du')' sign(u) + \int_0^\infty (\gamma u)' sign(u) = \int_0^\infty h sign(u), \\ &u'(0) sign(u(0)) = \frac{(\lambda + \widehat{\rho}_0)|u_0|}{b_0 - \gamma(0)} - \frac{h_0 sign(u(0))}{b_0 - \gamma(0)}. \end{aligned}$$

Since $u \in W^{2,1}_{loc}(\mathbb{R}_+) \subset \mathcal{C}^1(0, \infty)$, we get, for every finite $m > 0$

$$\int_0^m (du')' sign(u) \leq d(m)u'(m) sign(u(m)) - d(0)u'(0) sign(u(0))$$

and

$$\int_0^m (\gamma u)' sign(u) = \gamma(m)|u(m)| - \gamma(0)|u(0)|.$$

Consequently

$$\int_0^m (du')' sign(u) - \int_0^m (\gamma u)' sign(u) \leq (d(m)u'(m) - \gamma(m)u(m))\, sign(u(m)) + l_0$$

where $l_0 = -d(0)u'(0) sign(u(0)) + \gamma(0)|u(0)|$. Since

$$\lim_{m\to+\infty} d(m)u'(m) - \gamma(m)u(m) = 0$$

then

$$\begin{aligned} & \int_0^\infty (du')' sign(u) - \int_0^\infty (\gamma u)' sign(u) \\ = \quad & \lim_{m\to+\infty} \int_0^m (du')' sign(u) - \int_0^m (\gamma u)' sign(u) \leq l_0. \end{aligned}$$

Hence

$$\lambda \|u\|_{L^1} + \int_0^\infty \widehat{\mu}|u| \leq l_0 + \int_0^\infty h sign(u)$$

so

$$\lambda\|u\|_{L^1} + \int_0^\infty \widehat{\mu}|u| - \gamma(0)|u(0)| \le -d(0)u'(0)sign(u(0)) + \int_0^\infty hsign(u).$$

Since

$$d(0)u'(0)sign(u(0)) = \frac{d(0)(\lambda+\widehat{\rho}_0)|u_0|}{b_0-\gamma(0)} - \frac{d(0)h_0 sign(u(0))}{b_0-\gamma(0)}$$

then

$$\lambda\|u\|_{L^1} + \int_0^\infty \widehat{\mu}|u|ds + \left[-\gamma(0) + \frac{d(0)(\lambda+\widehat{\rho}_0)}{b_0-\gamma_0}\right]|u(0)| \le \frac{d(0)|h_0|}{b_0-\gamma(0)} + \|h\|_{L^1}$$

or

$$\lambda\|u\|_{L^1} + \int_0^\infty \widehat{\mu}|u|ds + \left[-\frac{\gamma(0)}{c_1} + (\lambda+\widehat{\rho}_0)\right]c_1|u(0)| \le \|h\|_{L^1} + c_1|h_0|.$$

Note that if

$$-\frac{\gamma(0)}{c_1} + \widehat{\rho}_0 \ge 0$$

then

$$\lambda\|u\|_{L^1} + \int_0^\infty \widehat{\mu}|u|ds + \lambda c_1|u(0)| \le \|h\|_{L^1} + c_1|h_0|.$$

Since

$$-\frac{\gamma(0)}{c_1} + \widehat{\rho}_0 = -\frac{\gamma(0)(b_0-\gamma(0))}{d(0)} + \gamma'(0) + \mu(0) + c_0 + \omega$$

is *nonnegative* for ω large enough then

$$\lambda\|u\|_{L^1} + \int_0^\infty (\mu+\omega)|u|ds + \lambda c_1|u(0)| \le c_1|h_0| + \|h\|_{L^1}$$

and

$$\lambda\|U\|_{\mathcal{X}_\infty} \le \|H\|_{\mathcal{X}_\infty}$$

1 for ω large enough. Finally $A_\infty - \omega I$ is dissipative.

4. Let us prove that $(\lambda I - A_\infty) : D(A_\infty) \to \mathcal{X}_\infty$ is a surjective operator for $\lambda > 0$ large enough. We consider first a particular case

$$H = (h, h_0)^T \in L^1(0,\infty) \cap L^2(0,\infty) \times \mathbb{R}$$

We look for $U = (u, u_0)^T \in D(A_\infty)$ such that $(\lambda I - A_\infty)U = H$, i.e.

$$(\lambda+\mu)u - (du')' + (\gamma u)' = h \text{ in } \mathbb{R}_+, \tag{29}$$

$$(\lambda+\rho_0)u_0 - (b_0-\gamma(0))u'(0) = h_0. \tag{30}$$

Multiply (29) by $v \in H^1(0,\infty)$ and integrate to get

$$\lambda\int_0^\infty uv + \int_0^\infty \mu uv - \int_0^\infty (du')'v + \int_0^\infty (\gamma u)'v = \int_0^\infty hv.$$

2 An integration by parts and (30) lead to

$$\lambda\int_0^\infty uv + \int_0^\infty \mu uv + \int_0^\infty du'v' - \int_0^\infty \gamma uv' + K_0 u(0)v(0) = \int_0^\infty hv + c_1 h_0 v(0). \tag{31}$$

One can show that the bilinear form defined by the left hand of (31) is coercive. By Lax-Milgram's Theorem, there exists a unique $u \in H^1(\mathbb{R}_+)$ satisfying (31) for all $v \in H^1(\mathbb{R}_+)$. It follows easily that $u \in H^2(\mathbb{R}_+)$. One sees that

$U = (u, u(0))$ satisfies (29)-(30). Since $u \in H^2(\mathbb{R}_+)$ then $u \in W^{2,1}(0,c)$ for every $c > 0$ and

$$\lim_{m\to\infty} u(m) = 0, \ \lim_{m\to\infty} u'(m) = 0.$$

Since $\gamma, d \in L^\infty(\mathbb{R}_+)$ then

$$\lim_{m\to+\infty} d(m)u'(m) - \gamma(m)u(m) = 0.$$

Let us prove that $u \in L^1(\mathbb{R}_+)$. Consider $\lambda := \tilde{\lambda} + \omega$, with $\tilde{\lambda}, \omega > 0$. Since

$$(\tilde{\lambda} + \widehat{\mu}(s))u(s) + (\gamma u)'(s) - (du')'(s) = h(s), s \in (0, \infty),$$
$$(\lambda + \widehat{\rho}_0)u_0 - (b_0 - \gamma(0))u'(0) = h_0,$$

then

$$\int_0^m (\tilde{\lambda} + \widehat{\mu}(s))\, |u(s)|\, ds = \int_0^m hsign(u) + \int_0^m (du')'sign(u) - \int_0^m (\gamma u)'sign(u)$$

and

$$u'(0)sign(u(0)) = \frac{(\tilde{\lambda} + \widehat{\rho}_0)|u_0|}{b_0 - \gamma(0)} - \frac{h_0 sign(u(0))}{b_0 - \gamma(0)}$$

so, using the above estimates,

$$(\tilde{\lambda} + \omega)\int_0^m |u(s)|\, ds$$
$$\leq \int_0^m hsign(u) + (d(m)u'(m) - \gamma(m)u(m))\, sign(u(m)) - d(0)u'(0)sign(u(0))$$
$$+\gamma(0)|u(0)|.$$

The fact that

$$\lim_{m\to+\infty} d(m)u'(m) - \gamma(m)u(m) = 0$$

gives

$$(\tilde{\lambda} + \omega)\int_0^{+\infty} |u(s)|\, ds \leq \int_0^{+\infty} hsign(u) - d(0)u'(0)sign(u(0)) + \gamma(0)|u(0)| < +\infty$$

and $u \in L^1(\mathbb{R}_+)$. Equation (29) shows that $(du')' - (\gamma u)' \in L^1(0,\infty)$. As for the previous finite case, by exploiting the *closedness* of A_∞, we get the surjectivity of

$$(\lambda I - A_\infty) : D(A_\infty) \to \mathcal{X}_\infty.$$

1 Finally A_∞ generates a C_0-semigroup $\{T_\infty(t)\}_{t\geq 0}$ by Lumer-Phillips' Theorem. □

Note that a priori the domain of the generator is *not*

$$\{(u, u_0) \in W^{2,1}(0,\infty) \times \mathbb{R} : u(0) = u_0\}$$

2 but this subspace turns out to be a core of $D(A_\infty)$. Indeed, we have:

Proposition 1. *Let $B : D(B) \subset \mathcal{X}_\infty \to \mathcal{X}_\infty$, be the restriction of A_∞ to*

$$\{(u, u_0) \in W^{2,1}(0,\infty) \times \mathbb{R} : u(0) = u_0\}.$$

3 *Then B is closable with closure A_∞.*

Proof. Note first that A_∞ is closed and

$$B \subset A_\infty$$

1 (in the sense of graphs) so $\overline{B} \subset A_\infty$ and $\overline{B}$ is a graph, i.e. B is closable.

To show that $\overline{B} = A_\infty$, it suffices to show that for any $U = (u, u(0)) \in D(A_\infty)$ there exists a sequence

$$U_n := (u^n, u^n(0)) \in D(B)$$

such that $u^n(0) \to u(0)$, $(u^n)'(0) \to u'(0)$,

$$u^n \to u \text{ in } L^1(\mathbb{R}_+)$$

2 and

$$(d\,(u^n)')' - (\gamma u^n)' \to (du')' - (\gamma u)' \text{ in } L^1(\mathbb{R}_+). \tag{32}$$

Let

$$\sigma : \mathbb{R} \to \mathbb{R}$$

be a C^2 function such that

$$\sigma(s) = \begin{cases} 1 \text{ for } s \le 0 \\ 0 \text{ for } s \ge 1. \end{cases}$$

Let

$$\sigma_n(s) := \sigma(s - n).$$

Note that

$$\sigma_n(s) = \begin{cases} 1 \text{ for } s \le n \\ 0 \text{ for } s \ge n + 1. \end{cases}$$

Let $U = (u, u(0)) \in D(A_\infty)$ and

$$u^n(s) := \sigma_n(s) u(s) \quad (s \ge 0).$$

Note that $u^n \in W^{2,1}(0, \infty)$ and $u^n = u$ on $[0, n]$. In particular $u^n(0) = u(0)$ and $(u^n)'(0) = u'(0)$. Since $\sigma_n(s) \le 1$ and

$$\lim_{n \to +\infty} \sigma_n(s) = 1 \ \forall s \ge 0$$

then

$$u^n \to u \text{ in } L^1(\mathbb{R}_+)$$

by the dominated convergence theorem. It suffices to show (32).
Note that

$$(u^n)' = \sigma_n' u + \sigma_n u'$$
$$(\gamma u^n)' = (\gamma \sigma_n u)' = (\gamma u)\, \sigma_n' + \sigma_n\, (\gamma u)'$$
$$d\,(u^n)' = d\sigma_n' u + d\sigma_n u'$$

and

$$(d\,(u^n)')' = \sigma_n''\,(du) + \sigma_n'\,(du)' + \sigma_n'\,(du') + \sigma_n\,(du')'$$

3 so

$$\begin{aligned} & (d\,(u^n)')' - (\gamma u^n)' \\ = \ & \sigma_n''\,(du) + \sigma_n'\,(du)' + \sigma_n'\,(du') + \sigma_n\,(du')' - (\gamma u)\,\sigma_n' - \sigma_n\,(\gamma u)' \\ = \ & \sigma_n \left[(du')' - (\gamma u)' \right] + \left[\sigma_n''\,(du) + \sigma_n'\,(d'u) - (\gamma u)\,\sigma_n' \right] + 2\sigma_n'\,(du')\,. \end{aligned}$$

Since $(du')' - (\gamma u)' \in L^1(\mathbb{R}_+)$ then

$$\sigma_n \left[(du')' - (\gamma u)' \right] \to (du')' - (\gamma u)'$$

in $L^1(\mathbb{R}_+)$ by the dominated convergence theorem. Note that

$$\sup_s |\sigma_n'(s)| = \sup_s |\sigma'(s)| < +\infty$$

$$\sup_s |\sigma_n''(s)| = \sup_s |\sigma''(s)| < +\infty$$

and the supports of σ_n' and σ_n'' are included in $[n, n+1]$ so

$$\sigma_n''(du) + \sigma_n'(d'u) - (\gamma u)\,\sigma_n' \to 0$$

in $L^1(\mathbb{R}_+)$ in $L^1(\mathbb{R}_+)$ by the dominated convergence theorem because du, $d'u$ and γu belong to $L^1(\mathbb{R}_+)$. The most tricky term is

$$\sigma_n'(du').$$

Since

$$\lim_{s \to +\infty} d(s)u'(s) - \gamma(s)u(s) = 0,$$

for any $\varepsilon > 0$ there exists $\overline{s} > 0$ such that

$$|d(s)u'(s) - \gamma(s)u(s)| \leq \varepsilon \quad (s \geq \overline{s}).$$

Then

$$|d(s)u'(s)| \leq \varepsilon + |\gamma(s)u(s)| \quad (s \geq \overline{s})$$

and

$$\begin{aligned}
\int_{\mathbb{R}_+} |\sigma_n'(s)d(s)u'(s)|\, ds &= \int_n^{n+1} |\sigma_n'(s)d(s)u'(s)|\, ds \\
&\leq \sup_s |\sigma'(s)| \int_n^{n+1} |d(s)u'(s)|\, ds \\
&\leq \varepsilon \sup_s |\sigma'(s)| + \sup_s |\sigma'(s)| \int_n^{n+1} |\gamma(s)u(s)|\, ds
\end{aligned}$$

(for n large enough) so

$$\limsup_{n \to +\infty} \int_{\mathbb{R}_+} |\sigma_n'(s)d(s)u'(s)|\, ds \leq \varepsilon \sup_s |\sigma'(s)|$$

since $\gamma u \in L^1(\mathbb{R}_+)$. Hence $\sigma_n'(du') \to 0$ in $L^1(\mathbb{R}_+)$ since ε is arbitrary. This ends the proof. □

3.3. **On irreducibility.**

The main result of this subsection is:

Proposition 2. *The C_0-semigroup $\{U_\infty(t)\}_{t \geq 0}$ is irreducible.*

Proof. As for the previous finite case, it suffices to prove that $(\lambda I - A_\infty)^{-1}$ is positivity improving. Let us show first that

$$(\lambda I - A_\infty)^{-1} \geq 0.$$

Let $U := (u, u_0) = (\lambda I - A_\infty)^{-1} H$ with $H = (h, h_0) \in \mathcal{X}_{\infty,+}$ and denote by $C_c^+([0,\infty[)$ the set of nonnegative continuous functions with compact support in $[0,\infty[$. Since $C_c^+([0,\infty[)$ is dense in $L_+^1(0,\infty)$ we may assume without loss of generality that

$$h \in C_c^+([0,\infty[).$$

Since $h \in (L^2 \cap L^1) \times \mathbb{R}$ then $u \in H^2(0,\infty)$. Now

$$(\lambda + \mu(s))u(s) + (\gamma u)'(s) - (du')'(s) = h(s), s \in (0,\infty),$$
$$(\lambda + \ \rho_0)u_0 - (b_0 - \gamma(0))u'(0) = h_0$$

shows that $u'' \in C(0,\infty)$. We write

$$-u'' + \rho_1 u' + \rho_2 u = \rho_3$$

where $\rho_1 = -(d' - \gamma)/d$,

$$\rho_2(s) = (\lambda + \mu(s) + \gamma'(s))/d(s) > 0 \ \forall s \text{ for } \lambda \text{ large enough}$$

and

$$\rho_3 = h/d \geq 0.$$

We want to show that $\inf u \geq 0$. If $\inf u < 0$ then the *absolute minimum* of u is achieved at some $\overline{s} \in [0,+\infty)$ since $\lim_{s\to+\infty} u(s) = 0$. This implies that $\overline{s} = 0$ otherwise

$$0 \geq -u''(\overline{s}) = -\rho_2(\overline{s})u(\overline{s}) + \rho_3(\overline{s}) \geq -\rho_2(\overline{s})u(\overline{s}) > 0$$

would lead to a contradiction. But if $\overline{s} = 0$ then $u(0) < 0$ and the boundary condition

$$(\lambda + \ \rho_0)u(0) - (b_0 - \gamma(0))u'(0) = h_0$$

gives

$$-(b_0 - \gamma(0))u'(0) = -(\lambda + \ \rho_0)u(0) + h_0 \geq -(\lambda + \ \rho_0)u(0) > 0$$

so $u'(0) < 0$ and then $u'(s) < 0$ in the neighborhood of $s = 0$ which contradicts the fact that the absolute minimum is achieved at 0. Hence

$$\inf u \geq 0.$$

Let us show now that $(\lambda I - A_\infty)^{-1}$ is *positivity improving*. As for the previous finite case, by using the resolvent identity, we may assume, without loss of generality, that

$$H \in D(A_\infty) \cap \mathcal{X}_+.$$

In particular $u'' \in C(0,\infty)$. Let us show that

$$u(s) > 0 \text{ a.e. and } u(0) > 0$$

once

$$H = (h, h_0) \in \mathcal{X}_{\infty,+} - \{0\}.$$

Let us show by contradiction that $u > 0$ everywhere. If the absolute minimum of u is *not* achieved, then $u > 0$ since $u \geq 0$. Consequently we only need to deal with the case where it is achieved at some $\overline{s} \in [0,\infty)$.

Suppose $u(\overline{s}) = 0$. Since $H \neq \{0\}$ then either $h_0 > 0$ or $\int_0^\infty h(s)ds > 0$. In any case, let $\overline{c} > \overline{s}$ such that ($h_0 > 0$ or $\int_0^{\overline{c}} h(s)ds > 0$). Note that the C^2 function

$$v := -u$$

satisfies the equation

$$v'' - \rho_1 v' + \widetilde{\rho}_2 v = h/d \geq 0$$

on $[0,\overline{c}]$, where $\widetilde{\rho}_2 \leq 0$. Note also that

$$\max_{[0,\overline{c}]} v = -\min_{[0,\overline{c}]} u \geq 0.$$

If u reaches its minimum in $(0,\overline{c})$ then v reaches its maximum in $(0,\overline{c})$. By the maximum principle (see [27] Theorem 3, p. 6), v must be constant and then u is equal to the constant $u(\overline{s})=0$. It follows that

$$h_0=0,\ h=0 \text{ on } [0,\overline{c}]$$

which is contradictory.
If v reaches its maximum (equal to zero) at $\overline{s}=0$ then $v'(0)<0$ by Hopf's maximum principle (see [27] Theorem 4, p. 7) which is contradictory since

$$(b_0-\gamma(0))v'(0)=h_0\geq 0.$$

Finally $u>0$ everywhere. □

3.4. **Asynchronous exponential growth.**

The main result of this subsection is:

Theorem 3.2. *We assume that $\beta_0(.)\neq 0$. Let there exist a measurable subset $I\subset\mathbb{R}_+$ with positive measure such that*

$$u\in L^1(\mathbb{R}_+),\ u(y)>0\ a.e. \Longrightarrow \int_0^\infty \beta(s,y)u(y)\mathrm{d}y>0\ a.e.\ s\in I. \tag{33}$$

If

$$\lim_{\lambda\to s(A_\infty)} r_\sigma(K_\infty(\lambda-A_\infty)^{-1})>1 \tag{34}$$

then the semigroup $\{U_\infty(t)\}_{t\geq 0}$ generated by $\mathcal{A}_\infty$ has asynchronous exponential growth.

Proof. Since A_∞ is resolvent positive and $K_\infty\geq 0$ then

$$K_\infty(\lambda-A_\infty)^{-1}\leq K_\infty(\mu-A_\infty)^{-1}\ (\lambda>\mu)$$

and

$$(s(A_\infty),\ +\infty)\ni\lambda\mapsto r_\sigma(K_\infty(\lambda-A_\infty)^{-1}) \tag{35}$$

is nonincreasing. Since $K_\infty(\lambda-A_\infty)^{-1}$ is weakly compact then $\left(K_\infty(\lambda-A_\infty)^{-1}\right)^2$ is compact (see e.g. [9] Corollary VI.13, p. 510). Note that

$$(s(A_\infty),\ +\infty)\ni\lambda\mapsto r_\sigma(K_\infty(\lambda-A_\infty)^{-1})$$

is convex and therefore continuous (see [20] p. 107). Assume *momentarily* that

$$r_\sigma(K_\infty(\lambda-A_\infty)^{-1})>0\ \ (\lambda>s(A_\infty)). \tag{36}$$

Then

$$(s(A_\infty),\ +\infty)\ni\lambda\mapsto r_\sigma(K_\infty(\lambda-A_\infty)^{-1})$$

is *strictly decreasing* (see [20] p. 106). If

$$\lim_{\lambda\to s(A_\infty)} r_\sigma(K_\infty(\lambda-A_\infty)^{-1})>1$$

then there exists a unique

$$\overline{\lambda}>s(A_\infty)$$

such that

$$r_\sigma(K_\infty(\overline{\lambda}-A_\infty)^{-1})=1.$$

Since $K_\infty(\overline{\lambda}-A_\infty)^{-1}$ is positive and power compact then

$$1=r_\sigma(K_\infty(\overline{\lambda}-A_\infty)^{-1})$$

is an isolated eigenvalue of $K_\infty(\overline{\lambda}-A_\infty)^{-1}$ associated to a nonnegative eigenfunction U so

$$K_\infty(\overline{\lambda}-A_\infty)^{-1}U = U.$$

Let

$$V := (\overline{\lambda}-A_\infty)^{-1}U.$$

Then $V \neq 0$ and

$$K_\infty V = K_\infty(\overline{\lambda}-A_\infty)^{-1}U = U = (\overline{\lambda}-A_\infty)V$$

so

$$\mathcal{A}_\infty V = \overline{\lambda}V.$$

As for the previous finite case, the weak compactness of K_∞ implies that $\{U_\infty(t)\}_{t\geq 0}$ and $\{T_\infty(t)\}_{t\geq 0}$ have the same essential type

$$\omega_{ess}(\mathcal{A}_\infty) = \omega_{ess}(A_\infty).$$

Since

$$\omega_{ess}(A_\infty) \leq s(A_\infty)$$

then

$$\omega_{ess}(\mathcal{A}_\infty) \leq s(A_\infty) < \overline{\lambda} = s(\mathcal{A}_\infty).$$

Thus $\{U_\infty(t)\}_{t\geq 0}$ exhibits a spectral gap and consequently $\{U_\infty(t)\}_{t\geq 0}$ has asynchronous exponential growth since it is irreducible. Finally, we have just to check (36). To this end, let $\overline{K} \in \mathcal{L}(\mathcal{X}_\infty)$ be defined by

$$\overline{K}\begin{pmatrix} u \\ u_0 \end{pmatrix} = \begin{pmatrix} \chi_I(s)\int_0^\infty \beta(s,y)u(y)dy \\ \int_0^\infty \beta_0(y)u(y)dy \end{pmatrix}.$$

where χ_I is the indicator function of I. Then

$$K(\lambda - A_\infty)^{-1} \geq \overline{K}(\lambda - A_\infty)^{-1}.$$

We identify $L^1(I)$ to the closed subspace of $L^1(\mathbb{R}_+)$ of functions vanishing a.e. outside I. Let

$$\mathcal{X}_\infty^I := L^1(I) \times \mathbb{R} \subset \mathcal{X}_\infty.$$

Since

$$\overline{K}(\lambda - A_\infty)^{-1} : \mathcal{X}_\infty \to \mathcal{X}_\infty^I$$

then

$$\overline{K}(\lambda - A_\infty)^{-1}_{|\mathcal{X}_\infty^I} : \mathcal{X}_\infty^I \to \mathcal{X}_\infty^I$$

and

$$K(\lambda - A_\infty)^{-1} \geq \overline{K}(\lambda - A_\infty)^{-1}_{|\mathcal{X}_\infty^I}$$

so

$$r_\sigma(K_\infty(\lambda - A_\infty)^{-1}) \geq r_\sigma(\overline{K}(\lambda - A_\infty)^{-1}_{|\mathcal{X}_\infty^I}).$$

Since $(\lambda - A_\infty)^{-1} : \mathcal{X}_\infty \to \mathcal{X}_\infty$ is positivity improving then our assumptions on β_0 and β imply that

$$\overline{K}(\lambda - A_\infty)^{-1}_{|\mathcal{X}_\infty^I} \; : \mathcal{X}_\infty^I \to \mathcal{X}_\infty^I$$

is positivity improving too. Since $\overline{K}(\lambda - A_\infty)^{-1}_{|\mathcal{X}_\infty^I}$ is weakly compact then $\left(\overline{K}(\lambda - A_\infty)^{-1}_{|\mathcal{X}_\infty^I}\right)^2$ is compact (see e.g. [9] Corollary VI.13, p. 510) and irreducible so

$$r_\sigma\left[\left(\overline{K}(\lambda - A_\infty)^{-1}_{|\mathcal{X}_\infty^I}\right)^2\right] > 0$$

(see e.g. [25] Theorem 3) and finally

$$r_\sigma\left[\overline{K}(\lambda - A_\infty)^{-1}_{|\mathcal{X}^I_\infty}\right] > 0.$$

This shows (36) and ends the proof.

□

Remark 6. Note that if

$$\lim_{\lambda \to s(A_\infty)} r_\sigma(K_\infty(\lambda - A_\infty)^{-1}) \leq 1$$

then $r_\sigma(K_\infty(\lambda - A_\infty)^{-1}) < 1 \ \ (\lambda > s(A_\infty))$ and

$$(\lambda I - \mathcal{A}_\infty)^{-1} = (\lambda I - A_\infty - K_\infty)^{-1} = (\lambda I - A_\infty)^{-1} \sum_{n=0}^{\infty} (K_\infty(\lambda I - A_\infty)^{-1})^n$$

$(\forall \lambda > s(A_\infty))$ shows that $s(\mathcal{A}_\infty) \leq s(A_\infty)$. In fact $s(\mathcal{A}_\infty) = s(A_\infty)$ since $s(\mathcal{A}_\infty) \geq s(A_\infty)$ due to $K_\infty \geq 0$.

Remark 7. Roughly speaking Theorem 3.2 expresses that $\{U_\infty(t)\}_{t\geq 0}$ has asynchronous exponential growth once $s(\mathcal{A}_\infty) > s(A_\infty)$. We mention that the spectral bound of generators of perturbed positive semigroups is characterized in [33] (see also [32]). Note that $s(\mathcal{A}_\infty)$ is not known explicitly. In case $s(A_\infty) = 0$, then (34) could be interpreted in terms of the basic reproduction number $\mathcal{R}_0$ (see [32]), we thank one of the referees for drawing our attention to this fact.

Remark 8. Note that $K_\infty(\lambda - A_\infty)^{-1}$ and $(\lambda - A_\infty)^{-1}K_\infty$ have the same non-zero spectrum (see e.g. [1] p. 196) and consequently

$$r_\sigma(K_\infty(\lambda - A_\infty)^{-1}) = r_\sigma((\lambda - A_\infty)^{-1}K_\infty).$$

On the other hand, $(\lambda - A_\infty)^{-1}K_\infty$ is never positivity improving since

$$K_\infty \begin{pmatrix} 0 \\ u_0 \end{pmatrix} = 0 \ \ \forall u_0 \in \mathbb{R}.$$

We end this subsection by a useful criterion to estimate a spectral radius.

Lemma 3.3. *Let*

$$\beta(x,y) = \beta_1(x)\beta_2(y)$$

where $\beta_1 \in L^1(0,\infty)$ *and* $\beta_2 \in L^\infty(0,\infty)$*. We assume that* β_1 *is continuous at* 0*. Then for every* $\lambda > s(A_\infty)$

$$r_\sigma\left(K_\infty(\lambda - A_\infty)^{-1}\right) = \left\| \beta_2 \left((\lambda - A_\infty)^{-1} \begin{pmatrix} \beta_1 \\ \beta_1(0) \end{pmatrix} \right)_1 \right\|_{L^1(\mathbb{R}_+)}.$$

Proof. We know that

$$\begin{aligned} K_\infty\left(\lambda - A_\infty\right)^{-1} \begin{pmatrix} f \\ f_0 \end{pmatrix} &= \begin{pmatrix} \beta_1(.) \left\| \beta_2 \left((\lambda - A_\infty)^{-1} \begin{pmatrix} f \\ f_0 \end{pmatrix} \right)_1 \right\|_{L^1} \\ \beta_1(0) \left\| \beta_2 \left((\lambda - A_\infty)^{-1} \begin{pmatrix} f \\ f_0 \end{pmatrix} \right)_1 \right\|_{L^1} \end{pmatrix} \\ &= \left\| \beta_2 \left((\lambda - A_\infty)^{-1} \begin{pmatrix} f \\ f_0 \end{pmatrix} \right)_1 \right\|_{L^1} \begin{pmatrix} \beta_1(.) \\ \beta_1(0) \end{pmatrix} \end{aligned}$$

so $K_\infty(\lambda - A_\infty)^{-1}$ is a one-rank operator with a single non-zero eigenvalue

$$\left\| \beta_2 \left((\lambda - A_\infty)^{-1} \begin{pmatrix} \beta_1(.) \\ \beta_1(0) \end{pmatrix} \right)_1 \right\|_{L^1}$$

associated to eigenvector

$$\begin{pmatrix} \beta_1(.) \\ \beta_1(0) \end{pmatrix}.$$

Hence

$$r_\sigma\left(K_\infty(\lambda - A_\infty)^{-1}\right) = \left\| \beta_2 \left((\lambda - A_\infty)^{-1} \begin{pmatrix} \beta_1 \\ \beta_1(0) \end{pmatrix} \right)_1 \right\|_{L^1(\mathbb{R}_+)}.$$

□

Remark 9. Note that if the kernel β is not separable but is bounded below by a separable kernel, i.e.

$$\beta(x,y) \geq \beta_1(x)\beta_2(y),$$

then a simple domination argument shows

$$r_\sigma\left(K_\infty(\lambda - A_\infty)^{-1}\right) \geq \left\| \beta_2 \left((\lambda - A_\infty)^{-1} \begin{pmatrix} \beta_1 \\ \beta_1(0) \end{pmatrix} \right)_1 \right\|_{L^1(\mathbb{R}_+)}.$$

Simplified models (with constant coefficients) are dealt with in [28] to check the property

$$\lim_{\lambda \to s(A_\infty)} \left\| \beta_2 \left((\lambda - A_\infty)^{-1} \begin{pmatrix} \beta_1 \\ \beta_1(0) \end{pmatrix} \right)_1 \right\|_{L^1(\mathbb{R}_+)} = +\infty.$$

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